Calculus II - Day 1

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1 Sequences

Definition: An ordered list of real numbers. $\{a_1, a_2, a_3, \ldots\}$, where $\ldots$ indicates the list goes on forever.

The sequence is given by:

$$\{a_n\}_{n=1}^{\infty} = \{a_n\}$$

The number a_n is called the n-th term in the sequence.

1.1 Example

The sequence $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$ is called the **Fibonacci sequence**.

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\} \xrightarrow{\text{defined by}} f_n = f_{n-2} + f_{n-1} \quad \text{for all } n \geq 3$$

with initial conditions $f_1 = 1$ *and* $f_2 = 1$.

The professor now asks, "Is there a formula for the golden ratio?"

A student responds, "There is a formula for the Fibonacci sequence:"

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

The professor explains, "If someone asks you for the 100th term in the sequence, you won't need the first 99 terms—only the formula."

A student asks, "Do you need to specify the domain of the formula?"

The professor responds, "Yes, the domain of the formula is $n \in \mathbb{Z}$ with $n \geq 1$."

1.2 Another Example

The sequence $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$ is presented.

A student suggests the following recurrence relation:

$$a_n = \frac{1}{2}a_{n-1} \quad \text{for } n \geq 2 \quad \text{and} \quad a_1 = 1$$

This is an example of a **geometric sequence**, which always has a formula.

1.3 Find Formulas for the Following Sequences

1.

$$\{a_n\}_{n=1}^{\infty} = \{2, 5, 8, 11, 14, \ldots\}$$

Solution + Rationale:
The first sequence increases by 3 each time, so it's defined by the recurrence relation:

$$f_n = f_{n-1} + 3 \quad \text{for } n \geq 1$$

For the plain non-function formula:

$$n = 3n - 1$$

Check:

$$n = 1: \quad 3(1) - 1 = 2$$
$$n = 2: \quad 3(2) - 1 = 5$$
$$n = 3: \quad 3(3) - 1 = 8$$

2.

$$\{b_n\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$$

Solution + Rationale:
This sequence is not based on the previous number in the sequence.
The second sequence is defined by the formula:

$$f_n = \frac{n}{n+1} \quad \text{with} \quad b_1 = \frac{1}{2}$$

The professor asks, "What kinds of questions can we ask about sequences?"
1. What happens as $n \to \infty$? Do the terms converge? Do they approach ∞ or $-\infty$? Something else?
2. Given two sequences $\{a_n\}$ and $\{b_n\}$ that are increasing, which one "grows faster"?
3. Is the sum of all the terms of the sequence finite?

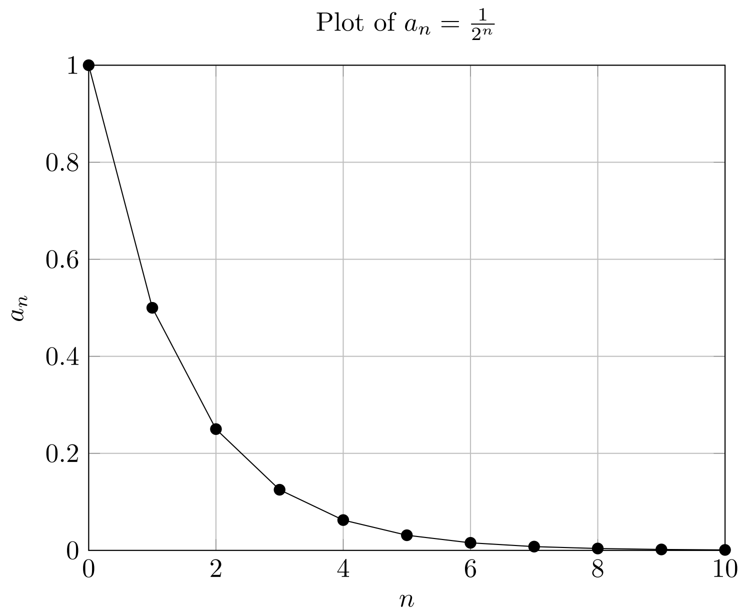
1.4 Definition

Definition: If the terms of a sequence $\{a_n\}$ approach a unique limit L as n increases, we say that L is the limit of the sequence, and that $\{a_n\}$ converges to L:

$$\lim_{n\to\infty} a_n = L$$

If the terms do not approach a single limit L as $n \to \infty$, we say that $\{a_n\}$ diverges.

Example: $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\} \quad \Rightarrow \quad a_n = \frac{1}{2^n}$



If $f(n)$ is a function and $\{a_n\}$ is a sequence such that $f(n) = a_n$ for all n, then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L$$

Here, $\frac{1}{2^x} \to 0$ as $x \to \infty$, so $\frac{1}{2^n} \to 0$ as $n \to \infty$.
Example:

$$a_n = \frac{n}{n+1} \quad \text{as} \quad n \to \infty, \quad a_n \to 1$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Professor notes that a useful tool is: L'Hopital's Rule.
Example:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = (1^\infty)$$

In this case, the limit is $e^1 = e$.

1.5 L'Hopital's Rule (Refresher)

L'Hopital's Rule:
If $\lim_{x\to c} \frac{f(x)}{g(x)}$ results in an indeterminate form like $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.
Back to the example. Let $L = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.
What is $\ln(L)$?

$$\ln(L) = \lim_{n\to\infty} \ln\left[\left(1 + \frac{1}{n}\right)^n\right] = \lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right)$$

$$\lim_{n\to\infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = \frac{0}{0}$$

Now, applying L'Hopital's Rule:

$$\lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-\frac{1}{x^2}}{\frac{1}{1+\frac{1}{x}} \cdot \frac{-1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x}} = 1$$

Thus, $\ln(L) = 1$, so $L = e^1 = \boxed{e}$.